Computers and Mathematics with Applications 59 (2010) 1944–1952

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Poincaré inequalities with the Radon measure for differential forms

Yuming Xing [a,*], Shusen Ding [b]

[a] Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, China

[b] Department of Mathematics, Seattle University, Seattle, WA 98122, USA

ARTICLE INFO

Article history:
Received 30 June 2009
Received in revised form 14 November 2009
Accepted 16 November 2009

Keywords:
Poincaré inequality
The A-harmonic equation
Differential forms

ABSTRACT

We establish the local and global Poincaré inequalities with the Radon measure for the solutions to the nonlinear elliptic partial differential equation for differential forms.

1. Introduction

The Poincaré inequalities have been playing an important role in analysis and related fields during the past several decades. The study and applications of Poincaré inequalities are now ubiquitous in different areas, including PDEs and potential analysis. Some versions of the Poincaré inequality with different conditions for various families of functions or differential forms have been developed in recent years. For example, in 1989, Susan G. Staples in [1] proved the following Poincaré inequality for Sobolev functions in L^s-averaging domains. If D is an L^p-averaging domain, $p \geq n$, then there exists a constant C, such that

$$\left(\frac{1}{m(D)}\int_D |u-u_D|^p \mathrm{d}m\right)^{1/p} \leq C(m(D))^{1/n}\left(\frac{1}{m(D)}\int_D |\nabla u|^p \mathrm{d}m\right)^{1/p}$$

for each Sobolev function u defined in D, where the integral is the Lebesgue integral, and $m(D)$ is the Lebesgue measure of D as appeared in [1], see [2–11] for more versions of the Poincaré inequality. However, most of these inequalities are developed with the Lebesgue measure. In this paper, we will establish the Poincaré inequalities with a more general measure, the Radon measure $\mu(x)$ which is induced by $\mathrm{d}\mu = g(x)\mathrm{d}x$, where $g(x)$ may be an unbounded function. For example, it is allowed that $g(x)$ contains a singular factor $\frac{1}{|x-x_0|^\alpha}$, here $\alpha > 0$ is a constant and x_0 is some fixed point in the integral domain. We are interested in the singular factor case because it is very often that we have to deal with the singular factor in applications, such as in the estimation of the Cauchy operator.

We assume that M is a domain in $\mathbb{R}^n$, $n \geq 2$, B and σB are the balls with the same center and $diam(\sigma B) = \sigma diam(B)$, $\sigma > 0$. We do not distinguish the balls from cubes, throughout this paper. We use $|E|$ to denote the Lebesgue measure of the set $E \subset \mathbb{R}^n$. Differential forms are extensions of functions in $\mathbb{R}^n$. For example, the function $u(x_1, x_2, \ldots, x_n)$ is called a 0-form. Moreover, if $u(x_1, x_2, \ldots, x_n)$ is differentiable, then it is called a differential 0-form. The 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^n u_i(x_1, x_2, \ldots, x_n)\mathrm{d}x_i$. If the coefficient functions $u_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, are differentiable, then $u(x)$

* Corresponding author.
E-mail addresses: xyuming@hit.edu.cn (Y. Xing), sding@seattleu.edu (S. Ding).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.11.012

is called a differential 1-form. Similarly, a differential k-form $u(x)$ is generated by $\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\}$, $k = 1, 2, \ldots, n$, that is, $u(x) = \sum_I u_I(x)dx_I = \sum u_{i_1 i_2 \cdots i_k}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, where $I = (i_1, i_2, \ldots, i_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, $D'(M, \wedge^l)$ be the space of all differential l-forms on M and $L^p(M, \wedge^l)$ be the l-forms $u(x) = \sum_I u_I(x)dx_I$ on M satisfying $\int_M |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We denote the exterior derivative by $d : D'(M, \wedge^l) \to D'(M, \wedge^{l+1})$ for $l = 0, 1, \ldots, n-1$, and define the Hodge star operator $\star : \wedge^k \to \wedge^{n-k}$ as follows. If $u = u_{i_1 i_2 \cdots i_k}(x_1, x_2, \ldots, x_n)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = u_I dx_I$, $i_1 < i_2 < \cdots < i_k$, is a differential k-form, then $\star u = \star(u_{i_1 i_2 \cdots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = (-1)^{\sum(I)} u_I dx_J$, where $I = (i_1, i_2, \ldots, i_k)$, $J = \{1, 2, \ldots, n\} - I$, and $\sum(I) = \frac{k(k+1)}{2} + \sum_{j=1}^k i_j$. The Hodge codifferential operator $d^\star : D'(M, \wedge^{l+1}) \to D'(M, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(M, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$. We write $\|u\|_{s,D} = \left(\int_D |u|^s\right)^{1/s}$ and $||u||_{s,D,\mu} = \left(\int_D |u|^s d\mu(x)\right)^{1/s}$, where $D \subset M$, and $\mu(x)$ is the Radon measure. We use $W^{1,p}(M, \wedge^k)$ to denote the Sobolev space of k-forms. For $0 < p < \infty$ and the Radon measure $\mu(x)$, the Sobolev norm with Radon measure of $u \in W^{1,p}(M, \wedge^k)$ over $D \subset M$ is denoted by

$$\|u\|_{W^{1,p}(D,\mu)} = (diam(D))^{-1}\|u\|_{p,D,\mu} + \|\nabla u\|_{p,D,\mu}. \tag{1.1}$$

We consider here the solutions to the nonlinear partial differential equation

$$d^\star A(x, du) = B(x, du) \tag{1.2}$$

which is called nonhomogeneous A-harmonic equation, where $A : M \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : M \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions: $|A(x, \xi)| \le a|\xi|^{p-1}$, $A(x, \xi) \cdot \xi \ge |\xi|^p$ and $|B(x, \xi)| \le b|\xi|^{p-1}$ for almost every $x \in M$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.2). A solution to (1.2) is an element of the Sobolev space $W^{1,p}_{loc}(M, \wedge^{l-1})$ such that $\int_M A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi = 0$ for all $\varphi \in W^{1,p}_{loc}(M, \wedge^{l-1})$ with compact support. If u is a function (0-form) in $\mathbb{R}^n$, the Eq. (1.2) reduces to

$$\operatorname{div} A(x, \nabla u) = B(x, \nabla u). \tag{1.3}$$

If the operator $B = 0$, Eq. (1.2) becomes $d^\star A(x, du) = 0$, which is called the (homogeneous) A-harmonic equation. Let $A : M \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x, \xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then, A satisfies the required conditions and $d^\star A(x, du) = 0$ becomes the p-harmonic equation $d^\star(du|du|^{p-2}) = 0$ for differential forms. If u is a function (0-form), the above equation reduces to the usual p-harmonic equation $\operatorname{div}(\nabla u|\nabla u|^{p-2}) = 0$ for functions. See [12–14,10,15,11] for recent results on the solutions to the different versions of the A-harmonic equation.

Let $D \subset \mathbb{R}^n$ be a bounded and convex domain. The linear operator $K_y : C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ was first introduced in [16], and then it was generalized to the following version in [17]. For any $y \in D$, there exists a linear operator $K_y : C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ defined by $(K_y\omega)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1}\omega(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1})dt$ and the decomposition $\omega = d(T\omega) + T(d\omega)$ holds. The homotopy operator $T : C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ is defined by $T\omega = \int_D \varphi(y)K_y\omega dy$, averaging K_y over all points y in D, where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y)dy = 1$. The l-form $\omega_D \in D'(D, \wedge^l)$ is defined by $\omega_D = |D|^{-1}\int_D \omega(y)dy$, $l = 0$, and $\omega_D = d(T\omega)$, $l = 1, 2, \ldots, n$, for all $\omega \in L^p(D, \wedge^l)$, $1 \le p < \infty$. From [17], we know that for any bounded and convex domain D, we have $\|\nabla(Tu)\|_{s,D} \le C|D|\|u\|_{s,D}$ and $\|Tu\|_{s,D} \le C|D|diam(D)\|u\|_{s,D}$. From [18, Page 16], any open subset Ω in $\mathbb{R}^n$ is the union of a sequence of cubes Q_k, whose sides are parallel to the axes, whose interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from F. More explicitly: (i) $\Omega = \cup_{k=1}^\infty Q_k$, (ii) $Q_j^0 \cap Q_k^0 = \phi$ if $j \ne k$, (iii) there exist two constants $c_1, c_2 > 0$ (we can take $c_1 = 1$, and $c_2 = 4$), so that $c_1 diam(Q_k) \le distance\ Q_k\ from\ F \le c_2 diam(Q_k)$. Hence, the definition of the homotopy operator T can be extended to any domain Ω in $\mathbb{R}^n$: For any $x \in \Omega$, $x \in Q_k$ for some k. Let T_{Q_k} be the homotopy operator defined on Q_k (each cube is bounded and convex). Thus, we can define the homotopy operator T_Ω on any domain Ω by $T_\Omega = \sum_{k=1}^\infty T_{Q_k}\chi_{Q_k}$. Hence, for any bounded domain G and any differential form $u \in L^s_{loc}(G, \wedge^l)$, we have

$$\|\nabla(Tu)\|_{s,G} \le C|G|\|u\|_{s,G}, \qquad \|Tu\|_{s,G} \le C|G|diam(G)\|u\|_{s,G}, \tag{1.4}$$

where C is a constant, independent of u, and $l = 1, 2, \ldots, n$, $1 < s < \infty$.

2. Local inequalities

We first introduce the following lemmas that will be used to prove the local Poincaré inequality with the Radon measure.

Lemma 2.1 ([8]). *Let u be a differential form satisfying the nonhomogeneous A-harmonic equation (1.2) in M, $\sigma > 1$ and $0 < s, t < \infty$. Then, there exists a constant C, independent of u, such that $\|du\|_{s,B} \le C|B|^{(t-s)/st}\|du\|_{t,\sigma B}$ for all balls or cubes B with $\sigma B \subset M$.*

Lemma 2.2. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbb{R}^n$, then*

$$\| fg \|_{s,E} \le \| f \|_{\alpha,E} \cdot \| g \|_{\beta,E} \tag{2.1}$$

for any $E \subset \mathbb{R}^n$.

Now, we prove the following local Poincaré inequality with the Radon measure which will be used to establish the global inequality in the next section.

Theorem 2.3. *Let $u \in L^s_{loc}(M, \wedge^l)$ be a solution of the nonhomogeneous A-harmonic equation (1.2) in a bounded domain M, $\mathrm{d}u \in L^s_{loc}(M, \wedge^{l+1})$, $l = 0, 1, \dots, n-1$ and $1 < s < \infty$. Then, for any ball B with $\sigma B \subset M$, there exists a constant C, independent of u, such that*

$$\left(\int_B |u - u_B|^s \mathrm{d}\mu\right)^{1/s} \le C|B|^\gamma \left(\int_{\sigma B} |\mathrm{d}u|^s \mathrm{d}\nu\right)^{1/s}, \tag{2.2}$$

where the Radon measures μ and ν are induced by $\mathrm{d}\mu = g(x)\mathrm{d}x$ and $\mathrm{d}\nu = h(x)\mathrm{d}x$, respectively, with $g, h \in L^1_{loc}(M)$, $0 < g(x) \le \frac{K_1}{|x-x_B|^\alpha}$ and $\frac{K_2}{|x-x_B|^\lambda} \le h(x)$. Here $K_1, K_2, \alpha, \lambda$ and σ are some constants with $K_1 > 0, K_2 > 0$, $n > \alpha > \lambda$, $\sigma > 1$, and $\gamma = 1 + \frac{1}{n} - \frac{\alpha-\lambda}{ns}$, x_B is the center of ball B.

Proof. Assume that $\varepsilon \in (0, 1)$ is small enough so that $\varepsilon n < \alpha - \lambda$ and $B \subset M$ is any ball with center x_B and radius r_B. Also, let $\delta > 0$ be small enough, $B_\delta = \{x \in B : |x - x_B| \le \delta\}$ and $D_\delta = B \setminus B_\delta$. For any differential forms u, we have $u = \mathrm{d}(Tu) + T(\mathrm{d}u) = u_B + T(\mathrm{d}u)$, where d is the exterior differential operator and T is the homotopy operator. From (1.4), we obtain

$$\|u - u_B\|_{s,B} = \|T(\mathrm{d}u)\|_{s,B} \le C_1|B| diam(B)\|\mathrm{d}u\|_{s,B}. \tag{2.3}$$

Since $0 < g(x) \le \frac{K_1}{|x-x_B|^\alpha}$, it follows that

$$\begin{aligned}\left(\int_{D_\delta} |u - u_B|^s \mathrm{d}\mu\right)^{1/s} &= \left(\int_{D_\delta} |u - u_B|^s g(x)\mathrm{d}x\right)^{1/s} \\ &\le \left(\int_{D_\delta} |u - u_B|^s \frac{K_1}{|x - x_B|^\alpha}\mathrm{d}x\right)^{1/s}.\end{aligned} \tag{2.4}$$

Choose $t = s/(1-\varepsilon)$, then $t > s$. Select $\beta = t/(t-s)$. By the Hölder inequality, (2.3) and (2.4), we obtain

$$\begin{aligned}\left(\int_{D_\delta} |u - u_B|^s \mathrm{d}\mu\right)^{1/s} &\le \left(\int_{D_\delta} \left(|u - u_B|\right)^s \frac{K_1}{|x - x_B|^\alpha}\mathrm{d}x\right)^{1/s} \\ &= \left(\int_{D_\delta} \left(|u - u_B| \frac{C_2}{|x - x_B|^{\alpha/s}}\right)^s \mathrm{d}x\right)^{1/s} \\ &\le \|u - u_B\|_{t,D_\delta} \left(\int_{D_\delta} \left(\frac{C_2}{|x - x_B|}\right)^{t\alpha/(t-s)} \mathrm{d}x\right)^{(t-s)/st} \\ &= \|u - u_B\|_{t,D_\delta} \left(\int_{D_\delta} C_3|x - x_B|^{-\alpha\beta}\mathrm{d}x\right)^{1/\beta s} \\ &\le \|u - u_B\|_{t,B} \cdot \||x - x_B|^{-\alpha}\|^{1/s}_{\beta,D_\delta} \\ &\le C_4|B| diam(B)\|\mathrm{d}u\|_{t,B} \cdot \||x - x_B|^{-\alpha}\|^{1/s}_{\beta,D_\delta}.\end{aligned} \tag{2.5}$$

We may suppose that $x_B = 0$. Otherwise, we can move the center to the origin by a simple transformation. Thus, for any $x \in B$, $|x - x_B| \ge |x| - |x_B| = |x|$. Using the polar coordinate substitution, we find that

$$\begin{aligned}\||x - x_B|^{-\alpha}\|^{1/s}_{\beta,D_\delta} &= \left(\int_{D_\delta} |x - x_B|^{-\alpha\beta}\mathrm{d}x\right)^{1/s\beta} \\ &\le \left(C_5 \int_\delta^{r_B} \rho^{-\alpha\beta}\rho^{n-1}\mathrm{d}\rho\right)^{1/s\beta} \\ &= \left|\frac{C_5}{n - \alpha\beta}(r_B^{n-\alpha\beta} - \delta^{n-\alpha\beta})\right|^{1/s\beta} \\ &\le C_6 \left|r_B^{n-\alpha\beta} - \delta^{n-\alpha\beta}\right|^{1/s\beta}.\end{aligned} \tag{2.6}$$

Set $m = nst/(ns + \alpha t - \lambda t)$, then $0 < m < s$. From Lemma 2.1, we have

$$\|du\|_{t,B} \le C_7 |B|^{\frac{m-t}{mt}} \|du\|_{m,\sigma B}, \tag{2.7}$$

where $\sigma > 1$ is a constant. Using the Hölder inequality again, we obtain

$$\begin{aligned}
\|du\|_{m,\sigma B} &= \left(\int_{\sigma B} \left(|du|(h(x))^{1/s}(h(x))^{-1/s}\right)^m dx\right)^{1/m} \\
&\le \left(\int_{\sigma B} |du|^s h(x) dx\right)^{1/s} \left(\int_{\sigma B} \left(\frac{1}{h(x)}^{1/s}\right)^{\frac{ms}{s-m}} dx\right)^{\frac{s-m}{ms}} \\
&\le \left(\int_{\sigma B} |du|^s h(x) dx\right)^{1/s} \left(\int_{\sigma B} \left(\frac{1}{h(x)}\right)^{\frac{m}{s-m}} dx\right)^{\frac{s-m}{ms}} \\
&\le \left(\int_{\sigma B} |du|^s h(x) dx\right)^{1/s} \left(\int_{\sigma B} \left(\frac{|x-x_B|^\lambda}{K_2}\right)^{\frac{m}{s-m}} dx\right)^{\frac{s-m}{ms}} \\
&\le \left(\int_{\sigma B} |du|^s h(x) dx\right)^{1/s} C_8 (\sigma r_B)^{\lambda/s + n(s-m)/ms} \\
&\le C_9 \left(\int_{\sigma B} |du|^s d\nu\right)^{1/s} (r_B)^{\lambda/s + n(s-m)/ms}.
\end{aligned} \tag{2.8}$$

By a simple calculation, we find that $n - \alpha\beta + \lambda\beta + n\beta(s-m)/m = 0$. Substituting (2.6)–(2.8) into (2.5) yields

$$\begin{aligned}
\left(\int_{D_\delta} |u-u_B|^s d\mu\right)^{1/s} &\le C_{10}|B|^{1+\frac{1}{n}+\frac{m-t}{mt}} \left(\int_{\sigma B} |du|^s d\nu\right)^{\frac{1}{s}} r_B^{\frac{\lambda}{s}+\frac{n(s-m)}{ms}} \left|r_B^{n-\alpha\beta} - \delta^{n-\alpha\beta}\right|^{1/s\beta} \\
&= C_{10}|B|^{1+\frac{1}{n}+\frac{m-t}{mt}} \left(\int_{\sigma B} |du|^s d\nu\right)^{\frac{1}{s}} \left[r_B^{(\frac{\lambda}{s}+\frac{n(s-m)}{ms})s\beta} \left|r_B^{n-\alpha\beta} - \delta^{n-\alpha\beta}\right|\right]^{1/s\beta} \\
&= C_{10}|B|^{1+\frac{1}{n}+\frac{m-t}{mt}} \left(\int_{\sigma B} |du|^s d\nu\right)^{\frac{1}{s}} \left|C_{11} r_B^{n-\alpha\beta+\lambda\beta+\frac{n\beta(s-m)}{m}} - \delta^{n-\alpha\beta} r_B^{\lambda\beta+\frac{n\beta(s-m)}{m}}\right|^{1/s\beta} \\
&\le C_{10}|B|^{1+\frac{1}{n}+\frac{m-t}{mt}} \left(\int_{\sigma B} |du|^s d\nu\right)^{\frac{1}{s}} \left[C_{11} r_B^{n-\alpha\beta+\lambda\beta+\frac{n\beta(s-m)}{m}} - \delta^{n-\alpha\beta} \delta^{\lambda\beta+\frac{n\beta(s-m)}{m}}\right]^{1/s\beta} \\
&\le C_{10}|B|^{1+\frac{1}{n}+\frac{m-t}{mt}} \left(\int_{\sigma B} |du|^s d\nu\right)^{\frac{1}{s}} \left[C_{11} r_B^{n-\alpha\beta+\lambda\beta+\frac{n\beta(s-m)}{m}} + \delta^{n-\alpha\beta+\lambda\beta+\frac{n\beta(s-m)}{m}}\right]^{1/s\beta} \\
&\le C_{12}|B|^{1+\frac{1}{n}-\frac{\alpha-\lambda}{ns}} \left(\int_{\sigma B} |du|^s d\nu\right)^{\frac{1}{s}} \\
&\le C_{12}|B|^{\gamma} \left(\int_{\sigma B} |du|^s d\nu\right)^{\frac{1}{s}},
\end{aligned}$$

that is,

$$\left(\int_{D_\delta} |u-u_B|^s d\mu\right)^{1/s} \le C_{12}|B|^{\gamma} \left(\int_{\sigma B} |du|^s d\nu\right)^{\frac{1}{s}}. \tag{2.9}$$

Notice that $\lim_{\delta\to 0}\left(\int_{D_\delta} |u-u_B|^s d\mu\right)^{1/s} = \left(\int_B |u-u_B|^s d\mu\right)^{1/s}$. Letting $\delta \to 0$ in (2.9), we obtain (2.2). The proof of Theorem 2.3 has been completed. □

Let $g(x) = \frac{1}{|x-x_B|^\alpha}$ and $h(x) = \frac{1}{|x-x_B|^\lambda}$ in Theorem 2.3, where α and λ are constants with $\alpha > \lambda$. We have the following version of the Poincaré inequality with the Radon measures.

Corollary 2.4. *Let $u \in L^s_{loc}(M, \wedge^l)$ be a solution of the nonhomogeneous A-harmonic equation (1.2) in a bounded domain M, $\mathrm{d}u \in L^s_{loc}(M, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$ and $1 < s < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|u - u_B\|_{s,B,\mu} \le C|B|^\gamma \|\mathrm{d}u\|_{s,\sigma B,\nu} \tag{2.10}$$

for all balls B with $\sigma B \subset M$, $\sigma > 1$, where the Radon measures μ and ν are induced by $\mathrm{d}\mu = g(x)\mathrm{d}x$ and $\mathrm{d}\nu = h(x)\mathrm{d}x$, respectively, with $g(x) = \frac{1}{|x-x_B|^\alpha}$ and $h(x) = \frac{1}{|x-x_B|^\lambda}$. Here α and λ are some constants with $\alpha > \lambda$, and $\gamma = 1 + \frac{1}{n} - \frac{\alpha-\lambda}{ns}$, x_B is the center of ball B.

Let $u \in L^s_{loc}(M, \wedge^0)$ be a solution of Eq. (1.3). From (1.1), we have

$$\|u - u_B\|_{W^{1,s}(B,\mu)} = (diam(B))^{-1}\|u - u_B\|_{s,B,\mu} + \|\nabla(u - u_B)\|_{s,B,\mu} \tag{2.11}$$

for any ball $B \subset M$. Note that $\|\nabla u\|_{s,B,\mu} = \|\mathrm{d}u\|_{s,B,\mu}$ and $\|\mathrm{d}(u_B)\|_{s,B,\mu} = 0$ now. Hence,

$$\|\nabla(u - u_B)\|_{s,B,\mu} = \|\mathrm{d}(u - u_B)\|_{s,B,\mu} \le \|\mathrm{d}u\|_{s,B,\mu} + \|\mathrm{d}u_B\|_{s,B,\mu} = \|\mathrm{d}u\|_{s,B,\mu}. \tag{2.12}$$

Substituting (2.12) into (2.11) and using (2.2) and the fact that $diam(B) = C_1|B|^{1/n}$ for some constant $C_1 > 0$, we have

$$\begin{aligned} \|u - u_B\|_{W^{1,s}(B,\mu)} &= C_2|B|^{-1/n}\|u - u_B\|_{s,B,\mu} + \|\mathrm{d}u\|_{s,B,\mu} \\ &\le C_3|B|^{-1/n}|B|^{1+1/n-(\alpha-\lambda)/ns}\|\mathrm{d}u\|_{s,\sigma_1 B,\nu} + \|\mathrm{d}u\|_{s,B,\mu} \\ &\le C_3|B|^{1-(\alpha-\lambda)/ns}\|\mathrm{d}u\|_{s,\sigma_1 B,\nu} + \|\mathrm{d}u\|_{s,B,\mu}. \end{aligned} \tag{2.13}$$

Using the same method as we developed in the proof of Theorem 2.3, we have

$$\|\mathrm{d}u\|_{s,B,\mu} \le C_4|B|^{(\lambda-\alpha)/ns}\|\mathrm{d}u\|_{s,\sigma_2 B,\nu}. \tag{2.14}$$

Combining (2.13) and (2.14) and noticing that M is bounded and $B \subset M$, we find that

$$\|u - u_B\|_{W^{1,s}(B,\mu)} \le (C_3|B| + C_4)|B|^{(\lambda-\alpha)/ns}\|\mathrm{d}u\|_{s,\sigma B,\nu} \le C_5|B|^{(\lambda-\alpha)/ns}\|\mathrm{d}u\|_{s,\sigma B,\nu},$$

where $\sigma = \max\{\sigma_1, \sigma_2\}$. Hence, we obtain the following Sobolev–Poincaré imbedding inequality with the Radon measure.

Corollary 2.5. *Let $u \in L^s_{loc}(M, \wedge^0)$ be a solution of Eq. (1.3) and all other conditions in Theorem 2.3 are satisfied. Then, there exists a constant C, independent of u, such that*

$$\|u - u_B\|_{W^{1,s}(B,\mu)} \le C|B|^{(\lambda-\alpha)/ns}\|\mathrm{d}u\|_{s,\sigma B,\nu}$$

for all balls B with $\sigma B \subset M$, $\sigma > 1$.

3. Global inequalities

In this section, we will prove the global Poincaré inequalities with the Radon measures. We first introduce the definition of John domains and the Covering Lemma.

Definition 3.1. A proper subdomain $\Omega \subset \mathbb{R}^n$ is called a δ-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that

$$d(\xi, \partial\Omega) \ge \delta|x - \xi|$$

for each $\xi \in \gamma$. Here $d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

Lemma 3.2 ([14] *Covering Lemma*). *Each Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that $\cup_i Q_i = \Omega$, $\sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q_i} \le N\chi_\Omega$ and some $N > 1$, and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R (this cube need not be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if Ω is δ-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q_m \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q_m$ from $\mathcal{V}$ and such that $Q_m \subset \rho Q_i$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$.*

Theorem 3.3. *Let Ω be any bounded and convex δ-John domain $\Omega \subset \mathbb{R}^n$ and $u \in L^s(\Omega, \wedge^0)$ be a solution of the nonhomogeneous A-harmonic equation (1.3),* $\mathrm{d}u \in L^s(\Omega, \wedge^1)$*, and $1 < s < \infty$. Then, there exists a constant C, independent of u, such that*

$$\left(\int_\Omega |u-u_{Q_0}|^s \mathrm{d}\mu\right)^{1/s} \le C|\Omega|^\gamma \left(\int_\Omega |\mathrm{d}u|^s \mathrm{d}\nu\right)^{1/s}, \tag{3.1}$$

where the Radon measures μ and ν are induced by $\mathrm{d}\mu = g(x)\mathrm{d}x$ and $\mathrm{d}\nu = h(x)\mathrm{d}x$, respectively, with $g, h \in L^1(\Omega)$, $0 < g(x) \le \frac{K_1}{d^\alpha(x,\partial\Omega)}$ and $\sum_i \chi_{Q_i}\frac{K_2}{|x-x_{Q_i}|^\lambda} \le h(x)$, x_{Q_i} is the center of Q_i with $\Omega = \cup_i Q_i$, and $\gamma = 1 + \frac{1}{n} - \frac{\alpha-\lambda}{ns}$. Here $K_1, K_2, \alpha, \lambda$ are some constants with $K_1, K_2 > 0$, $\lambda < \alpha < \min\{n, s+\lambda+n(s-1)\}$, and the fixed cube $Q_0 \subset \Omega$, the constant $N > 1$ and the cubes Q_i appeared in Lemma 3.2.

Proof. We may assume $g(x) \ge 1$ a.e. Otherwise, let $\Omega_1 = \Omega \cap \{x \in \Omega : 0 < g(x) < 1\}$ and $\Omega_2 = \Omega \cap \{x \in \Omega : g(x) \ge 1\}$, then $\Omega = \Omega_1 \cup \Omega_2$. We define the new function $G(x)$ by

$$G(x) = \begin{cases} 1, & x \in \Omega_1 \\ g(x), & x \in \Omega_2. \end{cases}$$

Also, we choose the constant $K_1 \ge diam^\alpha(\Omega)$, then $\frac{K_1}{d^\alpha(x,\partial\Omega)} \ge 1$ for any $x \in \Omega$. Therefore, $G(x) \ge g(x)$, and $G(x)$ satisfies all conditions required for $g(x)$, particularly, $0 < G(x) \le \frac{K_1}{d^\alpha(x,\partial\Omega)}$ and

$$\begin{aligned}\left(\int_\Omega |u-u_{Q_0}|^s \mathrm{d}\mu\right)^{1/s} &= \left(\int_\Omega |u-u_{Q_0}|^s g(x)\mathrm{d}x\right)^{1/s} \\ &\le \left(\int_\Omega |u-u_{Q_0}|^s G(x)\mathrm{d}x\right)^{1/s}\end{aligned} \tag{3.2}$$

with $G(x) \ge 1$. Hence, we may suppose that $g(x) \ge 1$ a.e. and have

$$\mu(Q) = \int_Q \mathrm{d}\mu = \int_Q g(x)\mathrm{d}x \ge \int_Q \mathrm{d}x = |Q|. \tag{3.3}$$

We use the notation appearing in Lemma 3.2. There is a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ for Ω such that $\Omega = \cup Q_i$, and $\sum_{Q_i\in\mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q_i} \le N\chi_\Omega$ for some $N > 1$. Since $\Omega = \cup Q_i$, for any $x \in \Omega$, it follows that $x \in Q_i$ for some i. It is easy to check that all conditions in Theorem 2.3 are satisfied. Applying Theorem 2.3 to Q_i, we obtain

$$\left(\int_{Q_i} |u-u_{Q_i}|^s \mathrm{d}\mu\right)^{1/s} \le C_1|Q_i|^\gamma \left(\int_{\sigma Q_i} |\mathrm{d}u|^s \mathrm{d}\nu\right)^{1/s}, \tag{3.4}$$

where $\sigma > 1$ is a constant. Using the elementary inequality $(a+b)^s \le 2^s(|a|^s + |b|^s)$, $s \ge 0$, we have

$$\begin{aligned}\left(\int_\Omega |u-u_{Q_0}|^s \mathrm{d}\mu\right)^{1/s} &= \left(\int_{\cup Q_i} |u-u_{Q_0}|^s \mathrm{d}\mu\right)^{1/s} \\ &\le \left(\sum_{Q_i\in\mathcal{V}}\left(2^s\int_{Q_i}|u-u_{Q_i}|^s\mathrm{d}\mu + 2^s\int_{Q_i}|u_{Q_i}-u_{Q_0}|^s\mathrm{d}\mu\right)\right)^{1/s} \\ &\le C_2(s)\left(\left(\sum_{Q_i\in\mathcal{V}}\int_{Q_i}|u-u_{Q_i}|^s\mathrm{d}\mu\right)^{1/s} + \left(\sum_{Q_i\in\mathcal{V}}\int_{Q_i}|u_{Q_i}-u_{Q_0}|^s\mathrm{d}\mu\right)^{1/s}\right)\end{aligned} \tag{3.5}$$

for a fixed $Q_0 \subset \Omega$. The first sum in (3.5) can be estimated by using Theorem 2.3 and the Covering Lemma

$$\begin{aligned}\left(\sum_{Q_i\in\mathcal{V}}\int_{Q_i}|u-u_{Q_i}|^s\mathrm{d}\mu\right)^{1/s} &\le \left(\sum_{Q_i\in\mathcal{V}} C_3|Q_i|^{s\gamma}\int_{\sigma Q_i}|\mathrm{d}u|^s\mathrm{d}\nu\right)^{1/s} \\ &\le \left(C_3|\Omega|^{s\gamma}\sum_{Q_i\in\mathcal{V}}\int_{\sigma Q_i}|\mathrm{d}u|^s\mathrm{d}\nu\right)^{1/s} \\ &\le C_4|\Omega|^\gamma\left(\int_\Omega|\mathrm{d}u|^s\mathrm{d}\nu\right)^{1/s}.\end{aligned} \tag{3.6}$$

We use the properties of δ-John domain to estimate the second sum in (3.5). Fix a cube $Q_m \in \mathcal{V}$ and let $Q_0, Q_1, \dots, Q_k = Q_m$ be the chain in Lemma 3.2.

$$|u_{Q_m} - u_{Q_0}| \le \sum_{i=0}^{k-1} |u_{Q_i} - u_{Q_{i+1}}|. \tag{3.7}$$

The chain $\{Q_i\}$ also has property that, for each i, $i = 0, 1, \dots, k-1$, with $Q_i \cap Q_{i+1} \neq \emptyset$, there exists a cube D_i such that $D_i \subset Q_i \cap Q_{i+1}$ and $Q_i \cup Q_{i+1} \subset ND_i$, $N > 1$.

$$\frac{max\{|Q_i|, |Q_{i+1}|\}}{|Q_i \cap Q_{i+1}|} \le \frac{max\{|Q_i|, |Q_{i+1}|\}}{|D_i|} \le C_5.$$

For such D_j, $j = 0, 1, \dots, k-1$, Let $|D| = min\{|D_0|, |D_1|, \dots, |D_{k-1}|\}$, then

$$\frac{max\{|Q_i|, |Q_{i+1}|\}}{|Q_i \cap Q_{i+1}|} \le \frac{max\{|Q_0|, |Q_1|, \dots, |Q_k|\}}{|D|} \le C_6. \tag{3.8}$$

By (3.3), (3.7), (3.8) and (3.4), we have

$$\begin{aligned}
|u_{Q_i} - u_{Q_{i+1}}|^s &= \frac{1}{\mu(Q_i \cap Q_{i+1})} \int_{Q_i \cap Q_{i+1}} |u_{Q_i} - u_{Q_{i+1}}|^s \mathrm{d}\mu \\
&\le \frac{1}{|Q_i \cap Q_{i+1}|} \int_{Q_i \cap Q_{i+1}} |u_{Q_i} - u_{Q_{i+1}}|^s \mathrm{d}\mu \\
&\le \frac{C_7}{max\{|Q_i|, |Q_{i+1}|\}} \int_{Q_i \cap Q_{i+1}} |u_{Q_i} - u_{Q_{i+1}}|^s \mathrm{d}\mu \\
&\le C_8 \sum_{j=i}^{i+1} \frac{1}{|Q_j|} \int_{Q_j} |u - u_{Q_j}|^s \mathrm{d}\mu \\
&\le C_9 \sum_{j=i}^{i+1} \frac{|Q_j|^{\gamma s}}{|Q_j|} \int_{\sigma Q_j} |\mathrm{d}u|^s \mathrm{d}\nu \\
&= C_9 \sum_{j=i}^{i+1} |Q_j|^{\gamma s - 1} \int_{\sigma Q_j} |\mathrm{d}u|^s \mathrm{d}\nu.
\end{aligned} \tag{3.9}$$

Since $Q_m \subset \rho Q_j$ for $j = i, i+1$, $0 \le i \le k-1$, and $\gamma s - 1 > 0$ when $\lambda < \alpha < min\{n, s + \lambda + n(s-1)\}$, using (3.9) we find that

$$\begin{aligned}
|u_{Q_i} - u_{Q_{i+1}}|^s \chi_{Q_m}(x) &\le C_{10} \sum_{j=i}^{i+1} \chi_{\rho Q_j}(x) |Q_j|^{\gamma s - 1} \int_{\sigma Q_j} |\mathrm{d}u|^s \mathrm{d}\nu \\
&\le C_{11} \sum_{j=i}^{i+1} \chi_{\rho Q_j}(x) |\Omega|^{\gamma s - 1} \int_{\sigma Q_j} |\mathrm{d}u|^s \mathrm{d}\nu \\
&\le C_{11} |\Omega|^{\gamma s - 1} \sum_{j=i}^{i+1} \chi_{\rho Q_j}(x) \int_{\sigma Q_j} |\mathrm{d}u|^s \mathrm{d}\nu.
\end{aligned} \tag{3.10}$$

Taking the sth root both sides in (3.10), and using $(a+b)^{1/s} \le 2^{1/s}(|a|^{1/s} + |b|^{1/s})$, (3.7) and (3.8), we obtain

$$|u_{Q_m} - u_{Q_0}| \chi_{Q_m}(x) \le C_{12} |\Omega|^{\gamma - 1/s} \sum_{D_i \in \mathcal{V}} \left(\int_{\sigma D_i} |\mathrm{d}u|^s \mathrm{d}\nu \right)^{1/s} \cdot \chi_{\rho D_i}(x) \tag{3.11}$$

for every $x \in \mathbb{R}^n$. Raising both sides of inequality (3.11) to s powers, and then integrating over $\mathbb{R}^n$ both sides, we have

$$\sum_{Q_m \in \mathcal{V}} \int_{Q_m} |u_{Q_m} - u_{Q_0}|^s \mathrm{d}\mu \le C_{13} |\Omega|^{\gamma s - 1} \int_{\mathbb{R}^n} \left| \sum_{D_i \in \mathcal{V}} \left(\int_{\sigma D_i} |\mathrm{d}u|^s \mathrm{d}\nu \right)^{1/s} \chi_{\rho D_i}(x) \right|^s \mathrm{d}\mu. \tag{3.12}$$

Notice that

$$\sum_{D_i \in \mathcal{V}} \chi_{\rho D_i}(x) \le \sum_{D_i \in \mathcal{V}} \chi_{\rho N D_i}(x) \le N \chi_{\Omega}(x).$$

Using elementary inequality $|\sum_{i=1}^{M} t_i|^s \le M^{s-1}\sum_{i=1}^{M}|t_i|^s$ for $s > 1$, we finally have

$$\begin{aligned}\left(\sum_{Q_m\in\mathcal{V}}\int_{Q_m}|u_{Q_m}-u_{Q_0}|^s\mathrm{d}\mu\right)^{1/s} &\le C_{14}|\Omega|^{\gamma-1/s}\left(\int_{\mathbb{R}^n}\left(\sum_{D_i\in\mathcal{V}}\left(\int_{\sigma D_i}|\mathrm{d}u|^s\mathrm{d}\nu\right)\chi_{\rho D_i}(x)\right)\mathrm{d}\mu\right)^{1/s}\\ &= C_{15}|\Omega|^{\gamma-1/s}\left(\sum_{D_i\in\mathcal{V}}\left(\int_{\sigma D_i}|\mathrm{d}u|^s\mathrm{d}\nu\right)\right)^{1/s}\\ &\le C_{16}|\Omega|^{\gamma}\left(\int_{\Omega}|\mathrm{d}u|^s\mathrm{d}\nu\right)^{1/s}. \end{aligned}\tag{3.13}$$

Substituting (3.6) and (3.13) in (3.5), we obtain (3.1). The proof of Theorem 3.3 has been completed. □

Similarly, choose $g(x) = \frac{1}{d^{\alpha}(x,\partial\Omega)}$ and $h(x) = \sum_i \chi_{Q_i}\frac{K}{|x-x_{Q_i}|^{\lambda}}$ in Theorem 3.3, where x_{Q_i} is the center of Q_i with $\Omega = \cup_i Q_i$, $\alpha > \lambda$ and $K > 0$. We have the following version of the Poincaré inequality with the Radon measures.

Corollary 3.4. *Let $u \in L^s(\Omega, \wedge^0)$ be a solution of the nonhomogeneous A-harmonic equation (1.3), $\mathrm{d}u \in L^s(\Omega, \wedge^1)$, $1 < s < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|u-u_{Q_0}\|_{s,\Omega,\mu} \le C|\Omega|^{\gamma}\|\mathrm{d}u\|_{s,\Omega,\nu}\tag{3.14}$$

for any bounded and convex δ-John domain $\Omega \subset \mathbb{R}^n$ and $\gamma = 1+\frac{1}{n}-\frac{\alpha-\lambda}{ns}$, where the Radon measures μ and ν are induced by $g(x) = \frac{1}{d^{\alpha}(x,\partial\Omega)}$ and $h(x) = \sum_i \chi_{Q_i}\frac{K}{|x-x_{Q_i}|^{\lambda}}$ respectively, where x_{Q_i} is the center of Q_i with $\Omega = \cup_i Q_i$, Here K, α, λ are constants with $K > 0$, $\lambda < \alpha < min\{n, s+\lambda+n(s-1)\}$, and the fixed cube $Q_0 \subset \Omega$, the constant $N > 1$ and the cubes Q_i appeared in Lemma 3.2.

Using (1.1) and (3.1) and noticing the fact that $|\Omega| < \infty$ since Ω is bounded, we have

$$\begin{aligned}\|u-u_{Q_0}\|_{W^{1,s}(\Omega,\mu)} &= (diam(\Omega))^{-1}\|u-u_{Q_0}\|_{s,\Omega,\mu}+\|\nabla(u-u_{Q_0})\|_{s,\Omega,\mu}\\ &\le (diam(\Omega))^{-1}C_1|\Omega|^{\gamma}\|\mathrm{d}u\|_{s,\Omega,\nu}+\|\mathrm{d}(u-u_{Q_0})\|_{s,\Omega,\mu}\\ &\le C_1|\Omega|^{1-(\alpha-\lambda)/ns}\|\mathrm{d}u\|_{s,\Omega,\nu}+\|\mathrm{d}u\|_{s,\Omega,\mu}\\ &\le C_2\|\mathrm{d}u\|_{s,\Omega,\nu}+\|\mathrm{d}u\|_{s,\Omega,\mu}.\end{aligned}\tag{3.15}$$

Thus, we have the following global Sobolev–Poincaré imbedding inequality with the Radon measure.

Corollary 3.5. *Assume that all conditions in Theorem 3.3 are satisfied. Then, there exists a constant C, independent of u, such that*

$$\|u-u_{Q_0}\|_{W^{1,s}(\Omega,\mu)} \le C\|\mathrm{d}u\|_{s,\Omega,\nu}+\|\mathrm{d}u\|_{s,\Omega,\mu}$$

for any bounded and convex δ-John domain $\Omega \subset \mathbb{R}^n$.

Acknowledgements

We would like to thank the referee for giving us thoughtful suggestions which greatly improved the presentation of the paper. The first author was supported by the NSF of China (# 10771044) and by Science Research Foundation at Harbin Institute of Technology (HITC200709). The second author was supported by the Bannan Chair Fund of College of Science and Engineering at Seattle University.

References

[1] S.G. Staples, L^p-averaging domains and the Poincaré inequality, Ann. Acad. Sci. Fenn, Ser. AI. Math. 14 (1989) 103–127.
[2] G. Acosta, R. Durán, An optimal Poincaré inequality in L^1 for convex domains, Proc. Amer. Math. Soc. 132 (2004) 195–202.
[3] R.P. Agarwal, S. Ding, C.A. Nolder, Inequalities for Differential Forms, Springer, September,2009.
[4] W. Beckner, A generalized Poincaré inequality for Gaussian measures, Proc. Amer. Math. Soc. 105 (1989) 397–400.
[5] M. Belloni, B. Kawohl, A symmetry problem related to Wirtinger's and Poincaré inequality, J. Differential Equations 156 (1999) 211–218.
[6] R.D. Benguria, C. Depassier, A reversed Poincaré inequality for monotone functions, J. Inequal. Appl. 5 (2000) 91–96.
[7] Chavel, E.A. Feldman, An optimal Poincaré inequality for convex domains of non-negative curvature, Arch. Ration. Mech. Anal. 65 (1977) 263–273.
[8] S. Ding, C.A. Nolder, Weighted Poincaré-type inequalities for solutions to the A-harmonic equation, Illinois J. Math. 2 (2002) 199–205.
[9] E. Hebey, Sharp Sobolev-Poincaré inequalities on compact Riemannian manifolds, Trans. Amer. Math. Soc. 354 (2002) 1193–1213.
[10] Y. Wang, C. Wu, Sobolev imbedding theorems and Poincaré inequalities for Green's operator on solutions of the nonhomogeneous A-harmonic equation, Comput. Math. Appl. 47 (2004) 1545–1554.
[11] Y. Xing, Weighted Poincaré-type estimates for conjugate A-harmonic tensors, J. Inequal. Appl. 1 (2005) 1–6.

[12] S. Ding, Two-weight Caccioppoli inequalities for solutions of nonhomogeneous *A*-harmonic equations on Riemannian manifolds, Proc. Amer. Math. Soc. 132 (2004) 2367–2375.
[13] B. Liu, $A_r^{\lambda}(\Omega)$-weighted imbedding inequalities for A-harmonic tensors, J. Math. Anal. Appl. 273 (2002) 667–676.
[14] C.A. Nolder, Hardy-Littlewood theorems for *A*-harmonic tensors, Illinois J. Math. 43 (1999) 613–631.
[15] Y. Xing, Weighted integral inequalities for solutions of the *A*-harmonic equation, J. Math. Anal. Appl. 279 (2003) 350–363.
[16] H. Cartan, Differential Forms, Houghton Mifflin Co., Boston, 1970.
[17] T. Iwaniec, A. Lutoborski, Integral estimates for null Lagrangians, Arch. Ration. Mech. Anal. 125 (1993) 25–79.
[18] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princton University Press, Princton, 1970.